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Reation between the roots and coefficients of polynomial Equation.

Relation between the roots & coefficients :→

$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$
has n roots,

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n roots,

$\therefore a_0x^n + a_1x^{n-1} + \cdots + a_n$
$\equiv a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

$\therefore a_0x^{n} + a_1x^{n-1} + \cdots + a_n$
$\equiv a_0\{x^n - (\alpha_1+\alpha_2+\alpha_3 \cdots + \alpha_n)x^{n-1} + (\alpha_1\alpha_2 + \alpha_1\alpha_3 \cdots \cdots)x^{n-2} - (\alpha_1\alpha_2\alpha_3 + \cdots)x^{n-3} + \cdots \cdots + (-1)^n\alpha_1\alpha_2\cdots\alpha_n\}$

$\equiv a_0\{x^n - (\Sigma\alpha_1)x^{n-1} + (\Sigma\alpha_1\alpha_2)x^{n-2} - (\Sigma\alpha_1\alpha_2\alpha_3)x^{n-3} + \cdots \cdots + (-1)^n\alpha_1\alpha_2\cdots\alpha_n\}$

$\therefore$ Comparing different coefficients of like powers of x from the two sides

$-(\Sigma\alpha_1)a_0 = a_1, \quad (-1)^2(\Sigma\alpha_1\alpha_2)a_0 = a_2, \ldots\ldots$

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$(-1)^3 (\sum \alpha_1 \alpha_2 \alpha_3) a_0 = a_3, \quad \ldots$

$- - - - - - - -$

$(-1)^r (\sum \alpha_1 \alpha_2, \cdots \alpha_r) a_0 = a_r,$

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$(-1)^n (\alpha_1 \alpha_2, \cdots \alpha_n) a_0 = a_n$

$$\therefore \sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1 \alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}, \ \cdots$$

$- - - -$

$$\sum (\alpha_1 \alpha_2, \cdots \alpha_r) = (-1)^r \frac{a_r}{a_0}$$

$- - - - -$

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$$\alpha_1 \alpha_2 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Ex - 1

Find the condition which must be satisfied by the coefficients of the equation $x^3 - px^2 + qx - r = 0$, when two of its roots α, β are ~~satisfied~~ connected by the relation $\alpha + \beta = 0$.

solution: →

$$x^3 - px^2 + qx - r = 0 \quad \alpha, \beta, \gamma$$

Where $\alpha + \beta = 0$

$$\therefore \alpha + \beta + \gamma = p$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = q,$$

$$\alpha\beta\gamma = r$$

$$\because \alpha + \beta = 0 \Rightarrow \gamma = p$$

$\because \gamma$ is a root of the equation

$$\gamma^3 - p\gamma^2 + q\gamma - r = 0$$

$$\therefore p^3 - p^3 + qp - r = 0$$

$$\Rightarrow pq = r$$

This is the required condition.

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(Q2) Solve $x^3-5x^2-2x+24=0$, when product of two roots $= \underline{12}$.

Solution:→

$$\alpha\cdot\beta = 12$$

$$\& \quad \alpha\cdot\beta\cdot\gamma = -24$$

$$\Rightarrow 12\cdot\gamma = -24$$

$$\Rightarrow \gamma = -2.$$

$\therefore (x+2)$ is a factor of $\underline{x^3-5x^2-2x+24=0}$.

$$\begin{array}{r|rrr|r}
-2 & 1 & -5 & -2 & 24 \\
 & & -2 & 14 & -24 \\
\hline
 & 1 & -7 & 12 & 0
\end{array}$$

$$\therefore x^3-5x^2-2x+24 = (x+2)(x^2-7x+12)$$
$$= (x+2)(x-3)(x-4)$$

$\therefore$ The roots of the equation are $-2, 3, 4$.

#

Example: → Solve the equation $x^3 - 9x^2 + 23x - 15 = 0$, the roots are being in A.P.

Solution: → The given equation is

$$x^3 - 9x^2 + 23x - 15 = 0 \quad \text{——— (1)}$$

Let its root in A.P. be $\alpha - \beta$, α, $\alpha + \beta$.

Then

$$\Sigma\alpha = 9$$

$$\Rightarrow \alpha - \beta + \alpha + \alpha + \beta = 9 \Rightarrow 3\alpha = 9 \Rightarrow \alpha = 3.$$

But α is a root of the equation (1), i.e; i.e; one root of (1) is 3.

Let us divide L.H.S of (1) by $(x-3)$ by synthetic division method we get.

$$\begin{array}{c|cccc} 3 & 1 & -9 & 23 & -15 \\ & & 3 & -18 & 15 \\ \hline & 1 & -6 & 5 & \multicolumn{1}{|c}{0} \end{array}$$

∴ The quotient is $x^2 - 6x + 5$

put the quotient equal to zero. we get

$$x^2 - 6x + 5 = 0$$

$$\Rightarrow (x-1)(x-5) = 0$$

$$\Rightarrow x = 1, 5$$

Hence, the roots of equation (1) are $\underline{1, 3, 5}$.

Example ② Solve the cubic equation $x^3 - 7x^2 + 14x - 8 = 0$, it being given that its roots are in G.P.

Solution:→

The given equation is

$$x^3 - 7x^2 + 14x - 8 = 0 \quad \text{—— (1)}$$

Let its three roots in G.P. be

$$\frac{\alpha}{\beta}, \alpha, \alpha\beta.$$

Then $\alpha\beta\gamma = 8$

$$\Rightarrow \frac{\alpha}{\beta} \cdot \alpha \cdot \alpha \cdot \beta = 8$$

$$\Rightarrow \alpha^3 = 8$$

$$\Rightarrow \alpha = 2$$

But α is a root of the equation (1), i.e; one root of (1) is 2.

Let us divide L.H.S of (1) by $x-2$ by ~~syn~~ synthetic division method. We get

2	1	−7	14	−8
		2	−10	8
	1	−5	4	

∴ The quotient is $x^2 - 5x + 4$.

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putting the quotient equal to zero, we get

$x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0$

$\Rightarrow x = 1, 4.$

Hence, the roots of equation ① are 1, 2, 4.

Example:→ Solve the equation $3x^3 + 11x^2 + 12x + 4 = 0$, being given that the roots are in H.P.

Solution:→ The given equation is

$3x^3 + 11x^2 + 12x + 4 = 0$ —— ①

Let the roots of equation ① be α, β, γ. Then

$S_1 = \alpha + \beta + \gamma = -\frac{11}{3}$ —— ②

$S_2 = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{12}{3} = 4$ —— ③

$S_3 = \alpha\beta\gamma = -\frac{4}{3}$ —— ④

Also, since α, β, γ are in H.P.

$$\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$$

$\Rightarrow \beta(\alpha+\gamma) = 2\alpha\gamma$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 3\alpha\gamma$ —— ⑤

from ③ & ⑤, we have

$$\frac{12}{3} = 3\alpha\gamma$$

$$\Rightarrow \alpha\gamma = \frac{4}{3} \qquad \text{——— ⑥}$$

Dividing ④ by ⑥, we get

$$\frac{\alpha\beta\gamma}{\alpha\gamma} = \frac{-4/3}{4/3} = -1$$

$$\Rightarrow \beta = -1$$

$\therefore$ One root of the equation ① is -1.

Let us divide the L.H.S of equation ① by $(x+1)$ by synthetic division

$$\begin{array}{r|rrrr} -1 & 3 & 11 & 12 & 4 \\ & & -3 & -8 & -4 \\ \hline & 3 & 8 & 4 & \vert\ 0 \end{array}$$

$\therefore$ The quotient is $3x^2+8x+4$

Put the quotient equal to zero, we get

$$3x^2+8x+4=0.$$

$$\Rightarrow x = \frac{-8 \pm \sqrt{64-48}}{6}$$

$$\Rightarrow x = \frac{-8 \pm 4}{6}$$

$$\Rightarrow x = \frac{-8+4}{6}, \frac{-8-4}{6}$$

$$\Rightarrow x = -\frac{2}{3}, -2$$

Hence the roots of the equation (1) are

$$-2, -1, -\frac{2}{3}.$$

Ans.

(Q1) Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, two of whose roots are in the ratio of $2:3$.

Solution:→ Let the roots of the equation be $2\alpha, 3\alpha, \beta$. Then we have

$$2\alpha + 3\alpha + \beta = 9 \Rightarrow 5\alpha + \beta = 9 \quad \text{--- (1)}$$

$$2\alpha \cdot 3\alpha + 2\alpha \cdot \beta + 3\alpha \cdot \beta = 14$$

$$\Rightarrow 5\alpha^2 + 5\alpha\beta = 14 \quad \text{--- (2)}$$

$$\& \; 2\alpha \cdot 3\alpha \cdot \beta = -24 \Rightarrow \alpha^2\beta = -4 \quad \text{--- (3)}$$

(5α) ① − ②, we get.

$$(5\alpha)[5\alpha+\beta] - 5\alpha^2 + 5\alpha\beta = 45\alpha - 14$$

$$\Rightarrow 19\alpha^2 - 45\alpha + 14 = 0$$

solving, we get

$$\alpha = \frac{45 \pm \sqrt{(45)^2 - 4\cdot 19\cdot 14}}{38}$$

$$= \frac{45 \pm 31}{38} = \frac{76}{38} \text{ or } \frac{14}{38} = 2 \text{ or } \frac{7}{19}$$

If $\alpha = 2$, then from ①

$$\beta = 9 - 5\alpha = 9 - 10 = -1.$$

If $\alpha = \frac{7}{19}$, then from ①

$$\beta = 9 - 5\alpha = 9 - \frac{35}{19} = \frac{136}{19}$$

But $\alpha = \frac{7}{19}$ & $\beta = \frac{136}{19}$ do not satisfy ③

Hence we take $\alpha = 2$ and consequently the roots are 4, 6 & −1.

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M.U.

Q1) Solve the equation $2x^3-15x^2+37x-30=0$ whose roots are in A.P.

Solution:→

Let $\alpha-\delta$, α, $\alpha+\delta$ be the roots of the given equation.

Then we have

$$(\alpha-\delta)+\alpha+(\alpha+\delta)=\frac{15}{2}$$

$$\Rightarrow 3\alpha=\frac{15}{2} \Rightarrow \alpha=\frac{5}{2} \quad \text{—— (1)}$$

Also, $(\alpha-\delta)\alpha+(\alpha-\delta)(\alpha+\delta)+\alpha(\alpha+\delta)=\frac{37}{2}$

$$\Rightarrow \alpha^2-\cancel{\alpha\delta}+\alpha^2-\delta^2+\alpha^2+\cancel{\alpha\delta}=\frac{37}{2}$$

$$\Rightarrow 3\alpha^2-\delta^2=\frac{37}{2} \Rightarrow 3\cdot\frac{25}{4}-\delta^2=\frac{37}{2}$$

$$\Rightarrow \delta^2=\frac{75}{4}-\frac{37}{2}=\frac{1}{4}$$

$$\therefore \delta=\pm\frac{1}{2}$$

$\therefore$ The roots are $\frac{5}{2}-\frac{1}{2}, \frac{5}{2}, \frac{5}{2}+\frac{1}{2}$

i.e; $2, \frac{5}{2}, 3$

Ans.

Remark:- If we take $\delta=-\frac{1}{2}$, we get the same roots.

Ex:→ Solve the equation $4x^4 - 28x^3 + 51x^2 - 7x - 20 = 0$ whose roots are in A.P.

solution:→

Let the roots of the given equation be $\alpha - 3\delta,\ \alpha - \delta,\ \alpha + \delta,\ \alpha + 3\delta.$

Then we have

$$(\alpha - 3\delta) + (\alpha - \delta) + (\alpha + \delta) + (\alpha - 3\delta) = \frac{28}{4} = 7 \quad -①$$

$$(\alpha - 3\delta)(\alpha - \delta) + \text{-----} = \frac{51}{4} \quad — ②$$

From ①, we have

$$4\alpha = 7 \Rightarrow \alpha = \frac{7}{4}$$

From ②, we have

$$6\alpha^2 - 10\delta^2 = \frac{51}{4}$$

$$\Rightarrow 10\delta^2 = 6\alpha^2 - \frac{51}{4}$$

$$= 6 \times \frac{49}{16} - \frac{51}{4} = \frac{45}{8}$$

$$\Rightarrow \delta^2 = \frac{45}{8 \times 10} = \frac{9}{16}.$$

$$\therefore \delta = \pm\frac{3}{4},$$

Taking the +ve sign of δ, we find that roots are

$$\frac{7}{4} - \frac{9}{4}, \frac{7}{4} - \frac{3}{4}, \frac{7}{4} + \frac{3}{4}, \frac{7}{4} + \frac{9}{4}$$

i.e., $-\frac{1}{2}, 1, \frac{5}{2}, 4.$

Ans.